

On pairs of locally convex spaces between which all operators are strictly singular

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ABSTRACT. A linear operator $T : X \rightarrow Y$ between topological vector spaces is called strictly singular if for any infinite dimensional vector subspace X_0 of X , the restriction of T to X_0 is not a topological isomorphism. In this note we investigate ordered pairs of locally convex spaces between which every linear continuous map is strictly singular. We give some sufficient conditions for such pairs in terms of possession of well-known properties on vector spaces. We also give some concrete examples and applications to topological tensor products. In particular, when one of the spaces is pre-Montel and has continuous norm, we obtain a complete characterization.

1. INTRODUCTION

Let X and Y be Banach spaces. An operator $T : X \rightarrow Y$ is called strictly singular ($T \in \mathbf{S}$), if for any infinite dimensional subspace X_0 of X ($X_0 \in \mathbb{I}(X)$), the restriction T_{X_0} is not a topological isomorphism. This concept was introduced by Kato [13] as a generalization of that of compact operators. Kato's definition was extended to locally convex spaces by van Dulst [28] who obtained a characterization of strictly singular operators on Ptak spaces. Wrobel [31] also characterized strictly singular operators on B_r -complete locally convex spaces into metrizable locally convex spaces.

After rising of the concept of strictly singular operators in Banach spaces, the question of whether there exists a sufficient condition under which any operator between two Banach spaces is strictly singular, is first suggested by Goldberg and Thorp [10] as a case in which they considered a pair of Banach spaces one of which is reflexive while the other contains no infinite dimensional reflexive subspace. We see another example of such an operator relation between Banach spaces in

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the note of Lacey and Whitley [16] in which they presented sufficient conditions under which any operator between Banach spaces is compact. It is mentioned that any operator from the space of summable sequences into a reflexive space is strictly singular. It is also shown that any operator from a weakly sequentially complete space to the space of null sequences is strictly singular.

In Section 3, we first discuss some conditions on pairs of Banach spaces between which all operators are compact which also take place in the ones which allow us to consider the relations between topological tensor products. Then in Section 4, we give sufficient conditions on pairs of Banach spaces between which all linear bounded maps are strictly singular, and some applications to the topological tensor products. Afterwards, in Section 5 we focus on locally convex setting. After dealing with the ideal property issues of strictly singular operators in locally convex spaces, we give some sufficient conditions so that any continuous linear operator between certain types of locally convex spaces is strictly singular.

2. PRELIMINARIES

A Banach space X is reflexive if and only if any bounded sequence in X has a weakly convergent subsequence in X . The space of all reflexive spaces forms a space ideal denoted by \mathbf{W} . A Banach space is called very-irreflexive if it does not contain any infinite dimensional reflexive subspace. A Banach space X is called quasi-reflexive ($X \in \mathbf{Q}$) if $\dim(X''/J(X)) < \infty$. A topological vector space X is called weakly conditionally compact, if any bounded sequence in X has a weakly Cauchy subsequence in X . In particular, if X is also a Banach space, it is called almost reflexive ($X \in \mathbf{R}$). By Rosenthal's ℓ^1 -Theorem [25], X contains no subspace isomorphic to ℓ^1 if and only if it is almost reflexive. A Banach space X is called weakly sequentially complete, if any weakly Cauchy sequence in X converges to an element weakly in X . A Banach space X is said to have the Schur property ($X \in \mathbf{V}$) if weak convergence and norm convergence are equivalent in X . Schur spaces are very-irreflexive. A Banach space is called a Grothendieck space if any weak* convergent sequence in X' converges weakly.

An operator $T : X \rightarrow Y$ is called (weakly)[$T \in \mathbf{W}$] compact ($T \in \mathbf{K}$) if for any bounded sequence $(x_n) \in X$, $(Tx_n) \in Y$ has a (weakly) convergent subsequence. An operator $T : X \rightarrow Y$ is called completely continuous ($T \in \mathbf{V}$), if for any weakly convergent sequence $(x_n) \in X$, (Tx_n) is a normed convergent sequence in Y . A property \mathcal{P} on a Banach space X is called hereditary if any infinite dimensional closed

subspace of X has \mathcal{P} . By \mathbf{X} , we denote the class of hereditarily- ℓ^1 Banach spaces. Reflexivity, Schur property, and subprojectivity [30] are well-known examples for hereditary properties. X is said to have nowhere \mathcal{P} if there is no infinite dimensional subspace of X having \mathcal{P} .

In Section 1, we defined the class of strictly singular operators. In general, there is no general interrelation between strictly singular, weakly compact, and completely continuous operators. Counterexamples might be found in [16, §II]. However, it can be easily verified that, any compact operator is strictly singular, while the converse is not true in general.

3. SUFFICIENT CONDITIONS FOR $(X, Y) \in \mathbf{K}$

For an advanced reading on the complete characterizations for relation \mathbf{K} we refer the reader to [22, 29, 33]. In Section 3 and 4, by an "operator" we mean a continuous linear mapping between Banach spaces. If every operator $T : X \rightarrow Y$ belongs to the operator ideal \mathbf{A} , then we write $(X, Y) \in \mathbf{A}$.

Theorem 3.1. [16, Corollary 6] *Let $X \in \mathbf{R}$ and let $Y \in \mathbf{V}$. Then, $(X, Y) \in \mathbf{K}$.*

Proof. Since X is almost reflexive, for any bounded sequence in X there exists a subsequence (x_n) with

$$\|x_{n_i} - x_{n_{i+1}}\| \xrightarrow{w} 0.$$

Consider $\|Tx_{n_i} - Tx_{n_{i+1}}\|$. Since $Y \in \mathbf{V}$, $T \in \mathbf{V}$. Therefore, the limit above is 0 in norm. That means (Tx_n) is Cauchy in Y . Since Y is complete, it is convergent. Therefore, $T \in \mathbf{K}$. \square

Theorem 3.2. *Let $X' \in \mathbf{V}$ and Y be weakly sequentially complete. Then, $(X, Y) \in \mathbf{K}$.*

Proof. By [16, Corollary 11], $(X, Y) \in \mathbf{W}$. Let $T \in (X, Y)$. Then, the conjugate map $T' \in L(Y', X')$ is also weakly compact. Then, T' maps bounded sequences in Y' into the sequences in X' which have weakly convergent subsequences. But $X' \in \mathbf{V}$, so those weakly convergent subsequences converge in norm. This makes T' compact. By the conjugacy property, T is also compact. Therefore, $(X, Y) \in \mathbf{K}$. \square

Theorem 3.3. *Let $X', Y \in \mathbf{V}$. Then, $(X, Y) \in \mathbf{K}$.*

Proof. By [18, 27], $L(X, Y)$ has Schur property. Now suppose there exists a non-compact operator $T \in L(X, Y)$. Since $Y \in \mathbf{V}$, T cannot be Rosenthal because if so, for every bounded sequence $(x_n) \in X$, there would exist a subsequence $(x_{k_n}) \in X$ such that $T(x_{k_n})$ is convergent.

So there exists a sequence $(\gamma_n) \in X$ with no weak Cauchy subsequence. That implies by Rosenthal's ℓ^1 -Theorem that $\ell^1 \hookrightarrow X$. However, by [6], $X \in \mathbf{R}$. Contradiction. Therefore $\mathbf{L}(X, Y) = \mathbf{K}(X, Y)$. \square

4. SUFFICIENT CONDITIONS FOR $(X, Y) \in \mathbf{S}$

Definition 4.1. A pair of Banach spaces (X, Y) is called *totally incomparable* if there exists no infinite dimensional Banach space Z which is isomorphic to a subspace of X and to a subspace of Y .

Theorem 4.2. If X and Y are totally incomparable, then $(X, Y) \in \mathbf{S}$.

Proof. Suppose $T \in \mathbf{L}(X, Y)$ is not strictly singular. Then, we may find $X_0 \in \mathbb{I}(X)$ to which T restricted is an isomorphism. This means $X_0 \cong T(X_0)$. But $T(X_0) \in \mathbb{I}(Y)$. Since X and Y are totally incomparable, this is impossible. \square

The following theorem, which is the first affirmative answer to the problem of existence of conditions under which all operators are strictly singular, is due to [10, Theorem (b)].

Theorem 4.3. Let X be very-irreflexive and $Y \in \mathbf{W}$. Then $(X, Y) \in \mathbf{S}$ and $(Y, X) \in \mathbf{S}$.

Our first result takes advantage of the idea of Theorem 4.3, and generalizes it.

Theorem 4.4. Let X be very-irreflexive, and $Y \in \mathbf{Q}$. Then, $(X, Y) \in \mathbf{S}$.

Proof. Suppose there exists a non-strictly singular $T : X \rightarrow Y$. Then, T is an isomorphism when restricted to $X_0 \in \mathbb{I}(X)$. Then, $X_0 \in \mathbf{Q}$. However, by [12, Lemma 2] there exists a reflexive subspace $X_1 \in \mathbb{I}(X_0)$. This contradicts very-irreflexivity of X . \square

Now we give the most general version of Theorem 4.3.

Theorem 4.5. Let \mathcal{P} be a property of Banach spaces which respects isomorphisms. Then, $(X, Y) \in \mathbf{S}$ and $(Y, X) \in \mathbf{S}$ if the following conditions are satisfied:

- (1) Y has hereditary \mathcal{P} ,
- (2) X has nowhere \mathcal{P} .

Proof. Let (X, Y) be a pair of Banach spaces satisfying (1) and (2), and for some $X_0 \in \mathbb{I}(X)$ suppose there exists $T : X \rightarrow Y$ such that $X_0 \cong T(X_0)$. But $T(X_0)$ inherits \mathcal{P} . Hence X_0 has \mathcal{P} . This contradicts (2). Now let $S : Y \rightarrow X$ be such that $Y_0 \cong S(Y_0) \subseteq X$ for some $Y_0 \in \mathbb{I}(Y)$. Since X has nowhere \mathcal{P} , $S(Y_0)$ does not enjoy \mathcal{P} . This contradicts (1). \square

Corollary 4.6. *Let $Y \in \mathbf{R}$, and $X \in \mathbf{X}$. Then $(X, Y) \in \mathbf{S}$.*

Corollary 4.7. *Let $X \in \mathbf{R}$. Then, $(X, \ell^1) \in \mathbf{S}$.*

Proof. Let $X_0 \in \mathbb{I}(X)$ on which T has a bounded inverse. So $X_0 \cong T(X_0) \in \mathbb{I}(\ell^1)$. By [17, Proposition 2.a.2], $T(X_0)$ contains a subspace Z with $Z \cong \ell^1$, so does M . By a note in [6], ℓ^1 can be embedded into X . Contradiction. \square

Lemma 4.8. [23, Proposition 1.6.3] *Let $T \in \mathbf{V}(X, Y)$. Then, for any weakly Cauchy sequence $(x_n) \in X$, $T(x_n) \in Y$ is norm convergent.*

Theorem 4.9. [16, Theorem 1.7] *Let $Y \in \mathbf{R}$. If $T \in \mathbf{V}(X, Y)$, then $T \in \mathbf{S}$.*

One may think of Theorem 4.9 as follows:

Theorem 4.10. *Let $X \in \mathbf{V}$ and $Y \in \mathbf{R}$. Then, $(X, Y) \in \mathbf{S}$.*

Proof. By Lemma 4.8 any operator T on X maps weakly Cauchy sequences into norm convergent sequences. That implies $T \in \mathbf{V}$. But $Y \in \mathbf{R}$. Hence by Theorem 4.9 the result follows. \square

Corollary 4.11. *Let $X', Y', Z' \in \mathbf{V}$ and let $W \in \mathbf{R}$. Then, $((X \hat{\otimes}_\pi Y)', W \hat{\otimes}_\pi Z) \in \mathbf{S}$.*

Proof. By Theorem 3.1, $(W, Z') \in \mathbf{K}$. So by [9, Theorem 3] we deduce $W \hat{\otimes}_\pi Z \in \mathbf{R}$. On the other hand, by [26, Theorem 3.3(b)] we reach that $L(X, Y') \in \mathbf{V}$. But in [27] it is proven that $L(X, Y') \cong (X \hat{\otimes}_\pi Y)'$. So $(X \hat{\otimes}_\pi Y)' \in \mathbf{V}$. Therefore, Theorem 4.10 yields the result. \square

A Banach space X is said to have the Dunford-Pettis property if any weakly compact operator with domain X is completely continuous.

Lemma 4.12. [6] *$X' \in \mathbf{V}$ if and only if X has the Dunford-Pettis property and $X \in \mathbf{R}$.*

Corollary 4.13. *Let X and Y belong to \mathbf{R} and let X have the Dunford-Pettis property. Then, $(X', Y) \in \mathbf{S}$.*

Theorem 4.14. *Let $X, Y \in \mathbf{R}$ and let Y have the Dunford-Pettis property. Then, $(X \hat{\otimes}_\pi Y, \ell^1) \in \mathbf{S}$.*

Proof. By Lemma 4.12, $Y' \in \mathbf{V}$. Then by Theorem 3.1, $(X, Y') \in \mathbf{K}$. Hence, [9, Theorem 3] yields that $X \hat{\otimes}_\pi Y \in \mathbf{R}$. So by Corollary 4.7, we are done. \square

As mentioned in [16], an almost reflexive Banach space has the nowhere Schur property. Hence, the following is straightforward in the light of Corollary 4.7.

Corollary 4.15. *Let X have the nowhere Schur property. Then, $(X, \ell^1) \in \mathbf{S}$.*

Lemma 4.16. [14, Theorem 2.1] *Infinite dimensional reflexive spaces have the nowhere Dunford-Pettis property.*

Theorem 4.17. *Let $X \in \mathbf{W}$, and $Y \in \mathbf{V}$. Then, $(X, Y) \in \mathbf{S}$.*

Proof. Let $T \in \mathbf{L}(X, Y)$ be such that on some $X_0 \in \mathbf{I}(X)$, T has a bounded inverse, that is, $X_0 \cong T(X_0)$. Since $Y \in \mathbf{V}$, Y has the hereditary Dunford-Pettis property [6]. Hence so does X_0 . But $X_0 \in \mathbf{W}$. By Lemma 4.16, this is a contradiction. \square

Theorem 4.18. *Let X and Y be reflexive spaces one of which having the approximation property, and let $(X, Y') \in \mathbf{K}$. Let W and Z be spaces having Schur property. Then, $(X \hat{\otimes}_\pi Y, W \check{\otimes}_\varepsilon Z) \in \mathbf{S}$.*

Proof. By [27, Theorem 4.21], $X \hat{\otimes}_\pi Y$ is reflexive. By [18], $W \check{\otimes}_\varepsilon Z$ has the Schur property. Then, Theorem 4.17 finishes the proof. \square

Theorem 4.19. *Let X be a Banach space having the hereditary Dunford-Pettis property and let $Y \in \mathbf{W}$. Then, $(X, Y) \in \mathbf{S}$.*

Proof. Let $X_0 \in \mathbf{I}(X)$ on which an arbitrary operator $T \in \mathbf{L}(X, Y)$ has a bounded inverse. Then, $X_0 \cong T(X_0)$. So, $X_0 \in \mathbf{W}$. So X_0 cannot have the Dunford-Pettis property. Contradiction. \square

Any operator T from any complemented subspace X_0 of each of $C(K)$, $B(S)$, $L_\infty(S, \Sigma, \mu)$, and $L(S, \Sigma, \mu)$ to a reflexive space is strictly singular [16].

Theorem 4.20. [16, Theorem 2.3] *Any weakly compact operator which is completely continuous, is strictly singular.*

Corollary 4.21. *Let $(X, Y) \in \mathbf{W}$ where X has the Dunford-Pettis property. Then, $(X, Y) \in \mathbf{S}$.*

Proof. Since X has the Dunford-Pettis property, $(X, Y) \in \mathbf{V}$. Then, by Theorem 4.20, T is strictly singular. \square

If any completely continuous operator with domain X is weakly compact, then X is said to have the reciprocal Dunford-Pettis property.

Corollary 4.22. *Let $(X, Y) \in \mathbf{V}$ where X has the reciprocal Dunford-Pettis property. Then, $(X, Y) \in \mathbf{S}$.*

Proof. Since X has the reciprocal Dunford-Pettis property and any $T : X \rightarrow Y$ is completely continuous, $T \in \mathbf{V} \cap \mathbf{W}$. By Theorem 4.20, we are done. \square

Theorem 4.23. *Let X be a Grothendieck space with the Dunford-Pettis property, and let Y be separable. Then, $(X, Y) \in \mathbf{S}$.*

Proof. By [21, Theorem 4.9], any operator $T : X \rightarrow Y$ is weakly compact. Since X has the Dunford-Pettis property, T is completely continuous. Hence, by Corollary 4.21, $(X, Y) \in \mathbf{S}$. \square

Example 1. Let K be a compact Hausdorff space and c_0 cannot be embedded into $C(K)$. Then, $C(K)$ is a Grothendieck space [24]. Hence, $(C(K), c_0) \in \mathbf{S}$.

Theorem 4.24. *Let $X \in \mathbf{R}$. Then, for any weakly sequentially complete Banach space Y , $(X, Y) \in \mathbf{W}$.*

Proof. Since $X \in \mathbf{R}$, if (x_n) is a bounded sequence in X , then (Tx_n) has a weakly Cauchy sequence in Y . But Y is weakly sequentially complete, that is, every weakly Cauchy sequence converges weakly in Y . Therefore, $T \in \mathbf{W}$. \square

Corollary 4.25. *Let $X \in \mathbf{R}$ be very-irreflexive and Y be weakly sequentially complete. Then, $(X, Y) \in \mathbf{S}$.*

Proof. By Theorem 4.24, $(X, Y) \in \mathbf{W}$. Now let $T : X \rightarrow Y$ which has a bounded inverse on $X_0 \in \mathbb{I}(X)$. If (x_n) is a bounded sequence in X_0 , then there exists (Tx_{k_n}) a weakly convergent subsequence of (Tx_n) in Y . Hence (x_{k_n}) is weakly convergent in X_0 , since T has a bounded inverse on X_0 . Thus, every bounded sequence in X_0 has a weakly convergent subsequence in X_0 . This is equivalent to saying that $X_0 \in \mathbf{W}$. Contradiction. \square

Example 2. Note that the non-reflexive space $c_0 \in \mathbf{R}$. Suppose there exists a reflexive subspace $E \in \mathbb{I}(c_0)$. Since $c_0 \notin \mathbf{W}$, it is not isomorphic to any subspace of E . But this is a contradiction to [17, Proposition 2.a.2].

A Banach space X is said to have the Dieudonné Property if for any Banach space Y , any operator $T \in \mathbf{L}(X, Y)$ mapping weakly Cauchy sequences into weakly convergent sequences is weakly compact.

Lemma 4.26. *Let X have the Dieudonné property and let Y be weakly sequentially complete. Then, $(X, Y) \in \mathbf{W}$.*

Proof. Let (x_n) be a weakly Cauchy sequence in X , and $T \in \mathbf{L}(X, Y)$. Then (Tx_n) is weakly Cauchy in Y . Since Y is weakly sequentially complete, (Tx_n) is weakly convergent. But X has the Dieudonné property, so $T \in \mathbf{W}$. \square

Theorem 4.27. *Let X possess both Dunford-Pettis and Dieudonné properties, and let Y be weakly sequentially complete. Then, $(X, Y) \in \mathbf{S}$.*

Proof. By Lemma 4.26, $(X, Y) \in \mathbf{W}$. Since X has the Dunford-Pettis property, $(X, Y) \in \mathbf{V}$. By Theorem 4.20, $(X, Y) \in \mathbf{S}$. \square

Example 3. $C(K)$, where K is a compact Hausdorff space, possesses both Dieudonné [11] and Dunford-Pettis [14] properties.

Lemma 4.28. $\mathbf{V} \subseteq \mathbf{X}$.

Proof. Let $X \in \mathbf{V}$ and $X_0 \in \mathbb{I}(X)$ with $X_0 \in \mathbf{R}$. Then, any bounded sequence (x_n) in X_0 , has a weakly Cauchy subsequence. X_0 inherits Schur property. Then the weakly Cauchy subsequence of (x_k) converges in X . Therefore, X_0 is finite dimensional. Contradiction. \square

5. STRICTLY SINGULAR RELATIONS BETWEEN LOCALLY CONVEX SPACES

Let \mathbf{P} be a class of Banach spaces having a certain property. Then, $\mathbf{s}(\mathbf{P})$ [4] is defined by the set of locally convex spaces X with local Banach spaces $X_U \in \mathbf{P}$ for which X_U is the completion of the normed space obtained by $X/p_U^{-1}(0)$, where U is an absolutely convex closed neighborhood of $\theta(X)$. In this section, by an "operator" we mean a linear map between locally convex spaces.

Definition 5.1. [28] *Let X and Y be locally convex spaces. A continuous linear operator $T : X \rightarrow Y$ is called strictly singular if it is not a topological isomorphism on any infinite dimensional linear subspace of its domain.*

In a locally convex space, the sum of two strictly singular operators need not be strictly singular [5]. Therefore, \mathbf{S} is not an operator ideal in the class of general locally convex spaces.

Definition 5.2. *An operator $T : X \rightarrow Y$ is called bounded, if T maps $U \in \mathcal{U}(X)$ to a bounded set in Y . The set of all bounded operators \mathbf{B} forms an operator ideal.*

$\mathbf{BS}(X, Y)$ denotes the intersection of $\mathbf{B}(X, Y)$ and $\mathbf{S}(X, Y)$. If $T \in \mathbf{BS}(X, Y)$, then T is said to be a bounded strictly singular operator from X into Y .

Proposition 5.3. [32, Proposition 1] *Let $T : X \rightarrow Y$ be a strictly singular operator where X and Y are Fréchet spaces. Then, either T is bounded or there is a continuous projection $P : Y \rightarrow Y$ such that $P(T(X)) = P(Y) \cong \omega$, where ω is the set of all sequences.*

By virtue of Proposition 5.3, we consider sufficient conditions on the pair of locally convex spaces (X, Y) such that any operator $T : X \rightarrow Y$ is bounded strictly singular.

Definition 5.4. [3] *A locally convex space X is called locally Rosenthal, if it can be written as a projective limit of Banach spaces each of which contains no isomorphic copy of ℓ^1 .*

A general locally convex space X with no copies of ℓ^1 need not to be locally Rosenthal. A counterexample might be found in [15]. In addition, one should assume that it is a quasinormable Fréchet space [19].

Lemma 5.5. [7, Lemma 2] *Let X and Y be locally convex spaces such that $X = \varprojlim X_k$ and $Y = \varprojlim Y_m$, where $\{X_k\}$ and $\{Y_m\}$ are collections of Banach spaces. If $(X_k, Y_m) \in \mathbf{S}$ for all m, k then every bounded operator on X into Y is strictly singular.*

Theorem 5.6. *Let X, Y be locally convex spaces where Y is locally Rosenthal, and $X \in \mathbf{s}(\mathbf{X})$. Then, every bounded operator on X into Y is strictly singular.*

Proof. Since Y is locally Rosenthal, there exists a family of Banach spaces $\{Y_m\}$ each of which does not contain an isomorphic copy of ℓ^1 such that $Y = \varprojlim Y_m$. Because $X \in \mathbf{s}(\mathbf{X})$, there exists a family of Banach spaces $\{X_k\}$ such that every $M_k \in \mathbb{I}(X_k)$ contains a subspace isomorphic to ℓ^1 . By Corollary 4.6, any linear operator $T_{mk} : X_k \rightarrow Y_m$ is strictly singular. Making use of Lemma 5.5, we deduce that every bounded operator $T : X \rightarrow Y$ is strictly singular. \square

Example 4. Let $\lambda_1(A) \in (d_2)$, and $\lambda_p(A) \in (d_1)$ as they are defined in [8]. Then, by [33], $(\lambda_1(A), \lambda_p(A)) \in \mathbf{B}$. It is known that $\lambda_p(A) = \varprojlim \ell^p(a_n)$, for $1 \leq p < \infty$. Since $\ell^p(a_n), 1 < p < \infty$ has no subspace isomorphic to ℓ^1 , $(\ell^1, \ell^p) \in \mathbf{S}$. Then, by Lemma 5.5, $(\lambda_1(A), \lambda_p(A)) \in \mathbf{BS}$.

Corollary 5.7. *Let Y be a quasinormable Fréchet space not containing an isomorphic copy of ℓ^1 , and let $X \in \mathbf{s}(\mathbf{V})$. Then, $(X, Y) \in \mathbf{BS}$.*

Proof. By [19, Theorem 6], Y is locally Rosenthal. Since $X \in \mathbf{s}(\mathbf{V})$, by Lemma 4.28, $X \in \mathbf{s}(\mathbf{X})$. Then, by Theorem 5.6, we are done. \square

Let X and Y be locally convex spaces. For a subspace X_0 of X , $\alpha \in I$ and $\beta \in J$, $\omega_{\alpha\beta}(T_{X_0}) := \sup\{q_\beta(Tx) : p_\alpha(x) \leq 1, x \in X_0\}$, where T_{X_0} is the restriction of T to X_0 . The following is a characterization of strictly singular operators in locally convex spaces.

Lemma 5.8. [20, Theorem 2.1] *Let $T : X \rightarrow Y$ be a continuous operator between locally convex spaces. Then T is strictly singular if and only if for any $\varepsilon > 0$, $\beta \in J$ and an infinite dimensional subspace X_0 of X there exists $\alpha_0 \in I$ and there exists an infinite dimensional subspace X_1 of X_0 such that $\omega_{\alpha\beta}(T_{X_1}) \leq \varepsilon$ for all α .*

Lemma 5.9. *$\mathcal{S}(X, Y)$ is an operator ideal if X is B_r -complete and Y is a metrizable locally convex space.*

Proof. Let $T : X \rightarrow Y$ and $S : X \rightarrow Y$ be strictly singular operators. Then, for any $X_0 \in \mathbb{I}(X)$, by [31, Theorem 1-IV], find $X_1 \in \mathbb{I}(X_0)$ such that T_{X_1} is precompact. Then find $X_2 \in \mathbb{I}(X_1)$ such that S_{X_2} is precompact. The ideal property of precompact operators yields the result. \square

The following theorem is an extension of [1, Problem 4.5.2] to B_r -complete locally convex spaces.

Theorem 5.10. *Let $X_i, i = 1, 2, \dots, n$ be B_r -complete and $Y_j, j = 1, 2, \dots, m$ be metrizable locally convex spaces, and let $\tau : \bigoplus_{i=1}^n X_i \rightarrow \bigoplus_{j=1}^m Y_j$ be a continuous operator. τ can be represented by uniquely determined continuous operators $T_{ji} : X_i \rightarrow Y_j$ so that $\tau = (T_{ji})_{i,j}$ is a matrix operator. Then, τ is strictly singular if and only if each T_{ji} is strictly singular.*

Proof. Let $X := \bigoplus_{i=1}^n X_i$ and $Y := \bigoplus_{j=1}^m Y_j$, for simplicity of notation, and assume that each T_{ji} is strictly singular. Let $\pi_i : X \rightarrow X_i$ be the canonical projection and define $\rho_j : Y_j \rightarrow Y$ by $\rho_j y_j = 0 \oplus 0 \oplus \dots \oplus 0 \oplus y_j \oplus 0 \oplus \dots \oplus 0$, for which y_j is the j -th summand. Consider $X \xrightarrow{\pi_i} X_i \xrightarrow{T_{ji}} Y_j \xrightarrow{\rho_j} Y$, and write $\tau_{ji} = \rho_j \circ T_{ji} \circ \pi_i$. Then $\tau_{ji}(x_1 \oplus x_2 \oplus \dots \oplus x_n) = 0 \oplus 0 \oplus \dots \oplus 0 \oplus T_{ji}x_i \oplus 0 \oplus \dots \oplus 0$, where T_{ji} is the j -th summand. By Lemma 5.9 and rewriting $\tau = \sum_{i=1}^n \sum_{j=1}^m \tau_{ji}$, τ is strictly singular.

For the converse, suppose that the continuous operator $\tau : X \rightarrow Y$ is strictly singular, and assume that the operator T_{ji} is not strictly singular for some i, j . Then by Lemma 5.8, for any infinite dimensional subspace M of X_i and for some $\beta \in J$, $\omega_{\alpha\beta}(\tau_N) > \varepsilon$, for all $\alpha \in I$. If we write $\hat{M} := \{0\} \oplus \{0\} \oplus \dots \oplus M \oplus \{0\} \oplus \dots \oplus \{0\}$ where M places in the i -th summand, \hat{M} is an infinite dimensional vector subspace of X . $\omega_{\alpha\beta}(\tau_{\hat{M}}) > \varepsilon$, for all $\alpha \in I$. But, that contradicts the assumption τ is strictly singular. \square

Theorem 5.11. *Let (X, Y) be a pair of locally convex spaces satisfying the following*

- (1) *Every infinite dimensional closed subspace of X contains a subspace isomorphic to X .*
- (2) *Y has no subspace isomorphic to X .*

Then, every continuous operator $T : X \rightarrow Y$ is strictly singular.

Proof. Assume that there exists an infinite dimensional closed subspace X_0 of X with T_{X_0} is a topological isomorphism. Then, by (1), $T_{X_0}(X_0) \subseteq Y$ contains a subspace which is isomorphic to X . But that contradicts (2). \square

Let X and Y be Fréchet spaces and let Y have continuous norm. Then, (2) in Theorem 5.11 is also necessary if every bounded subset of Y is precompact. Let any continuous operator $T : X \rightarrow Y$ be strictly singular. Then, by Theorem 5.3, it is bounded. Now let there exist $Y_0 \in \mathbb{I}(Y)$ which is isomorphic to X . Then $I_{Y_0} : Y_0 \rightarrow X$ is bounded. Then Y_0 is finite dimensional. Contradiction.

Example 5. Let $\Lambda_1(\alpha)$ and $\Lambda_\infty(\alpha)$ denote power series spaces of finite type and infinite type, respectively. By [33] we know that no subspace of $\Lambda_\infty(\alpha)$ can be isomorphic to a power series space of finite type. Choose α as it is in the proof (b) of [2, Theorem 1] so that any subspace X of $\Lambda_1(\alpha)$ with a basis has a complemented subspace which is isomorphic to a power series space of finite type. By Theorem 5.11, every continuous operator $T : \Lambda_1(\alpha) \rightarrow \Lambda_\infty(\alpha)$ is strictly singular.

REFERENCES

- [1] Abramovich YA, Aliprantis CD. Problems in Operator Theory. Graduate Studies in Mathematics. American Mathematical Society, 2002.
- [2] Aytuna A, Terzioğlu T. On certain subspaces of a nuclear power series spaces of finite type. *Studia Math* 1980; 69: 79–86.
- [3] Boyd C, Venkova M. Grothendieck space ideals and weak continuity of polynomials on locally convex spaces. *Monatsch Math* 2007; 151: 189–200.
- [4] Castillo JMF, Simões MA. Some problems for suggested thinking in Fréchet space theory. *Extr Math* 1991; 6: 96–114.
- [5] Dierolf S. A note on strictly singular and strictly cosingular operators. *Indag Math* 1981; 84: 67–69.
- [6] Diestel J. A survey of results related to the Dunford-Pettis property. *Contemp Math* 1980; 2: 15–60.
- [7] Djakov PB, Önal S, Terzioğlu T, Yurdakul M. Strictly singular operators and isomorphisms of Cartesian products of power series spaces. *Arch Math* 1998; 70: 57–65.
- [8] Dragilev MM. On regular basis in nuclear spaces. *Math Sbornik* 1965; 68: 153–175.
- [9] Emmanuelle G. Banach spaces in which Dunford-Pettis sets are relatively compact. *Arch Math* 1992; 58: 477–485.

- [10] Goldberg S, Thorp EO. On some open questions concerning strictly singular operators. *Proc Amer Math Soc* 1963; 14: 224–226.
- [11] Grothendieck A. Sur les applications lineares faiblement compactes d'espaces du type $C(K)$. *Canad J Math* 1953; 5: 129–173.
- [12] Herman R, Whitley RJ. An example concerning reflexivity. *Studia Math* 1967; 28: 289–294.
- [13] Kato T. Perturbation theory for nullity, deficiency and other quantities of linear operators. *J Analyse Math* 1958; 6: 261–322.
- [14] Kester MC. The Dunford-Pettis property. Ph.D. thesis, Oklahoma State University, 1972.
- [15] Köthe G. *Topological Vector Spaces I*. Springer-Verlag, 1969.
- [16] Lacey E, Whitley RJ. Conditions under which all the bounded linear maps are compact. *Math Ann* 1965; 58: 1–5.
- [17] Lindenstrauss J, Tzafriri L. *Classical Banach Spaces I and II*. Springer-Verlag, 1996.
- [18] Lust F. Produits tensoriels injectifs d'espaces de sidon. *Colloq Math* 1975; 32: 286–289.
- [19] Miñarro MA. A characterization of quasinormable Köthe sequence spaces. *Proc Amer Math Soc* 1995; 123: 1207–1212.
- [20] Moorthy CG, Ramasamy CT. Characterizations of strictly singular and strictly discontinuous operators on locally convex spaces. *Int Journ of Math Analysis* 2010; 4: 1217–1224.
- [21] Morrison TJ. *Functional Analysis: An Introduction to Banach Space Theory*. Wiley & Sons, 2000.
- [22] Nurlu Z. On pairs of Köthe spaces between which all operators are compact. *Math Nachr* 1985; 122: 277–287.
- [23] Pietsch A. *Operator Ideals*. North Holland, 1980.
- [24] Rübiger F. *Beiträge zur Strukturtheorie der Grothendieck-Räume*. Springer-Verlag, 1985.
- [25] Rosenthal HP. A characterization of Banach spaces not containing ℓ^1 . *Proc Nat Acad Sci USA* 1974; 71: 2411–2413.
- [26] Ryan R. The Dunford-Pettis property and projective tensor products. *Bull Polish Acad Sci Math* 1987; 35: 785–792.
- [27] Ryan R. *Introduction to the tensor products of Banach spaces*. Springer-Verlag, 2002.
- [28] van Dulst D. Perturbation theory and strictly singular operators in locally convex spaces. *Studia Math* 1970; 38: 341–372.
- [29] Vogt D. Fréchet Räume zwischen denen jede stetige Abbildung beschränkt ist. *J Reine Angew Math* 1983; 345: 182–200.
- [30] Whitley RJ. Strictly singular operators and their conjugates. *Trans Amer Math Soc* 1964; 113: 252–261.
- [31] Wrobel VV. Strikt singuläre Operatoren in lokalkonvexen Räumen. *Math Nachr* 1978; 83: 127–142.
- [32] Yurdakul M. A remark on a paper of J. Prada. *Arch Math* 1993; 61: 385–390.
- [33] Zahariuta V. On the isomorphism of Cartesian products of locally convex spaces. *Studia Math* 1973; 46: 201–221.

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